Exponential dist

- $Y \sim exp(\beta) = gamma(\underset{=1}{\alpha}, \beta)$

$$f_Y(y) = \frac{e^{-\frac{y}{\beta}}}{\beta}$$

If $U \sim$ Poisson$(1/\beta)$, time until next event is $T \sim exp(\beta)$

$$P(Y > t_1 + t_2 \mid Y > t_1) = P(Y > t_2)$$

Proof:

$$P(Y > a+b \mid Y > a) \stackrel{?}{=} P(Y > b)$$

$$= \frac{P(Y > a+b \text{ and } Y > a)}{P(Y > a)}$$

$$= \frac{P(Y > a+b)}{P(Y > a)}$$

$$P(Y > a+b) = \int_{a+b}^{\infty} \frac{e^{-\frac{y}{\beta}}}{\beta}\, dy$$

$$= \frac{1}{\beta} \times -\beta\, e^{-\frac{y}{\beta}} \Big|_{a+b}^{\infty}$$

$$= -e^{-\frac{\infty}{a+b}} + e^{-\frac{a+b}{\beta}} = e^{-\frac{a+b}{\beta}}$$

... same with $P(Y > a) = e^{-\frac{a}{\beta}}$

$$= \frac{e^{-\frac{a+b}{\beta}}}{e^{-\frac{a}{\beta}}} = e^{-\frac{b}{\beta}} = P(Y > b)$$

B - distribution

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

- $Y \sim beta(\alpha, \beta)$
- $f_Y(y) = \frac{y^{a-1}(1-y)^{\beta-1}}{B(a,\beta)}$ $\quad 0 < y < 1 \quad$ else 0

Model variable between c, d

$$U = \frac{Y-c}{d-c}$$

$$E[Y] = \frac{a}{a+\beta}$$

$$var[Y] = \frac{a\beta}{(a+\beta)^2(a+\beta+1)}$$

If $\alpha, \beta \in \mathbb{N} \setminus \{0\}$

$$P(Y<y) = \sum_{i=a}^{a+\beta-1} \binom{a+\beta-1}{i} y^i (1-y)^{\alpha+\beta-1-i}$$

$$E[Y] = \int_0^1 y \cdot \frac{y^{a-1}(1-y)^{\beta-1}}{B(a,\beta)} dy$$

$$= \frac{1}{B(a,\beta)} \cdot \int_0^1 y^a (1-y)^{\beta-1} dy$$

let $k-1 = a$
$k = a+1$

$$= \frac{B(k,\beta)}{B(a,\beta)} \cdot \int_0^1 \frac{y^{k-1}(1-y)^{\beta-1}}{B(k,\beta)} dy$$

p.d.f. over full range

$$= \frac{B(a+1,\beta)}{B(a,\beta)} \times 1$$

$$= \frac{\Gamma(a+1)\cancel{\Gamma(\beta)}}{\Gamma(a+\beta+1)} \times \frac{\Gamma(a+\beta)}{\Gamma(a)\cancel{\Gamma(\beta)}}$$

$$= \frac{\Gamma(a+1)\Gamma(a+\beta)}{\Gamma(a+\beta+1)\Gamma(a)} = \frac{\alpha\cancel{\Gamma(a)}\cancel{\Gamma(a+\beta)}}{(a+\beta)\cancel{\Gamma(a+\beta)}\cancel{\Gamma(\alpha)}}$$

$Var(Y) = \mu_2' - \mu^2$
$= E[Y^2] - E[Y-$

Moments

k-th non-central moment $\mu_k' = E(Y^k)$

k-th central moment $\mu_k = E((Y-\mu)^k)$

- Unique to each prob. distribution (all moments together)

Moment-generating fn

$m(t) = E[e^{tY}]$ exists if $m(t)$ is finite for $-b \leq t \leq b$

$= \sum_y e^{ty} p(y)$ or $\int_{-\infty}^{\infty} e^{ty} f_Y(y)\, dy$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Discrete

$$m(t) = E(e^{ty}) = \sum_y p(y) \times \left[1 + ty + \frac{(ty)^2}{2!} + \frac{(ty)^3}{3!} + \cdots\right]$$

$$= \sum_y p(y) + t\sum_y y p(y) + \frac{t^2}{2!} \sum_y p(y) y^2 + \frac{t^3}{3!} \sum_y p(y) y^3 + \cdots$$

non-central moment $k = E[Y^k]$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots$$

$$\mu_k' = \frac{d^k}{dt^k} m(t) \Big|_{t=0} = \mu_k' + t\mu_{k+1}' + \frac{t^2}{2!}\mu_{k+2}' + \cdots$$

Example: Poisson

$$Y \sim \text{Poisson}(\lambda)$$

$$m(t) = E[e^{tY}] = \sum_y e^{ty} \frac{\lambda^y e^{-\lambda}}{y!}$$

$$= e^{-\lambda} \cdot \underbrace{\sum_y \frac{(\lambda e^t)^y}{y!}}_{= e^{\lambda e^t}}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= e^{\lambda e^t - \lambda} = e^{\lambda(e^t - 1)}$$

Continuous var:

$$Y \sim \text{gamma}(\alpha, \beta)$$

$$m(t) = E[e^{tY}] = \int e^{ty} \times \frac{y^{\alpha-1} e^{-\frac{y}{\beta}}}{\beta^\alpha \Gamma(\alpha)} \, dy$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot \int y^{\alpha-1} e^{ty - \frac{y}{\beta}} \, dy$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot \int y^{\alpha-1} e^{y(t - \frac{1}{\beta})} \, dy$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int y^{\alpha-1} e^{\frac{y}{\frac{1}{t - \frac{1}{\beta}}}} \, dy$$

let $k = -\frac{\beta}{\beta t - 1}$

$$= \frac{\left(-\frac{\beta}{\beta t - 1}\right)^\alpha \Gamma(\alpha)}{\beta^\alpha \Gamma(\alpha)} \times \underbrace{\int \frac{y^{\alpha-1} e^{-\frac{y}{k}}}{k^\alpha \Gamma(\alpha)} \, dy}_{=1}$$

$$= \left(-\frac{1}{\beta t - 1}\right)^\alpha$$

$$= (1 - \beta t)^{-\alpha} \qquad t < \frac{1}{\beta}$$

Normal dist

$Y \sim normal(\mu, \sigma^2)$

$$m_Y(t) = \int e^{ty} \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$$

Easier sol: $X = Y - \mu \sim normal(0, \sigma^2)$

$$m_X(t) = \int e^{tx} \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int e^{tx - \frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx$$

$$tx - \frac{x^2}{2\sigma^2} = - \frac{-2\sigma^2 tx - x^2}{2\sigma^2} =$$

TODO: finish!

Tchebyshell's Thm

$Y \sim (\mu, \sigma^2)$ ← any dist

$$P(\mu - k\sigma < Y < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

N.B.

Proof

$$\sigma^2 = \int_{-\infty}^{\infty} (y-\mu)^2 f_Y(y)\,dy$$

all 3 ⊕ b.c. $(y-\mu)^2$ is ⊕ and f_Y is ⊕

$$= \int_{-\infty}^{\mu-k\sigma} \cdots dy + \int_{\mu-k\sigma}^{\mu+k\sigma} \cdots dy + \int_{\mu+k\sigma}^{\infty}$$

$$\geq \int_{-\infty}^{\mu-k\sigma} \cdots dy + \int_{\mu+k\sigma}^{\infty} \cdots dy$$

$$\int_{-\infty}^{\mu-k\sigma} (y-\mu)^2 f_Y(y)\,dy$$

$$y \leq \mu - k\sigma$$

$2 \geq -5$

$4 \geq 25$

$$y - \mu \leq -k\sigma$$

$$\mu - y \geq k\sigma$$

$$(\mu - y)^2 \geq (k\sigma)^2$$

$$(\mu-y)^2 f_Y(y) \geq (k\sigma)^2 f_Y(y)$$

$$\int_{-\infty}^{\mu-k\sigma} (\mu-y)^2 f_Y(y)\,dy \geq \int_{-\infty}^{\mu-k\sigma} (k\sigma)^2 f_Y(y)\,dy$$

$-4 \leq 2$

$16 \not\leq 4$